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THE APPROXIMATE CALCULATION OF A CLASS OF INTEGRO-DIFFERENTIAL EXPRESSIONS DESCRIBING THE PROPAGATION OF ELASTIC WAVES†

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A definition of a fractional derivative in the Marchaut form [1] is used to regularize the integro-differential expressions describing the propagation of elastic waves in solids. Using a spline interpolation, a stable algorithm is constructed for the approximate calculation of the initial integro-differential expressions, and estimates are obtained for the residues of the quadrature formulae.

THE SOLUTIONS of a wide range of problems in the dynamic theory of elasticity are unstable to small variations of both the region [2] and the boundary conditions [3]. Solutions in quadratures (see, for example, [3–7]) turn out to be unsuitable for practical calculations because they are represented in the form of integro-differential expressions with singularities. As is well known [8], numerical differentiation of functions whose values are found approximately with a certain error is an unstable operation.

1. FORMULATION OF THE PROBLEM

We will consider the integro-differential operator B acting on a function f of two independent variables as given by the formula

$$Bf(x, y, t)=\frac{\partial}{\partial t}\int_0^t d\tau\int_0^\infty g(x; y-\xi)\,\frac{(t-\tau)H(t-\tau-c^{-2}\sqrt{x^2+(y-\xi)^2})}{\sqrt{(t-\tau)^2-c^{-2}(x^2+(y-\xi)^2)}}f(\xi,\tau)d\xi \tag{1.1}$$

where H is the Heaviside function, $g(x, \alpha)$ is a fairly smooth function of two independent variables (which play the role of a weighting function), and c is a positive constant.

The expression on the right-hand side of (1.1) describes the propagation of elastic waves with a velocity c in a solid occupying a region $(0<x,\ 0<y)$ in a Cartesian system of coordinates. The function f defines the external action on the elastic solid from the side of one of the boundaries $(x=0)$, while B is a resolving operator corresponding to the mixed problem for the vector Lamé equation. Operators of the form (1.1) are encountered in many investigations connected with the reflection and diffraction of acoustic and elastic waves [5, 9]. Moreover, Green's functions of the initial and initial boundary-value problems for partial differential equations of the hyperbolic type are, as a rule, distributed. Hence, the construction of the solutions of such problems leads to the problem of regularizing the corresponding integro-differential expressions with movable singularities and the development of appropriate numerical methods.

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2. REGULARIZATION OF THE INTEGRO-DIFFERENTIAL EXPRESSION (1.1)

We will assume that the first derivative of the function f is of Hölder order $\alpha > \frac{1}{2}$. Replacing the variables $\xi = y \pm \sqrt{(c^2t^2v^2 - x^2)}$, $\tau = t - ts$ on the right-hand side of (1.1) and using the expression

$$s(s^2 - v^2)^{-\frac{1}{2}} = 1 + \int_s^\infty v^2(\alpha^2 - v^2)^{-\frac{3}{2}} d\alpha$$

we will have, after differentiation by parts and some reduction

$$Bf(x, y, t) = H(t - c^{-1}x) \left\{ \int_{v_0}^{w} \Psi_+(v)\Psi_-(v)dv + \int_{v_0}^{1} \Psi_-(v)\Psi_+(v)dv \right\} \tag{2.1}$$

Here

$$w = \min\left\{1, \sqrt{x^2+y^2}/(ct)\right\}, \quad v_0 = x/(ct), \quad \Psi_\pm(v) = c^2t^2vR^{-1}g(x, \pm R)$$

$$R = \sqrt{c^2t^2v^2 - x^2}, \quad \Phi_\pm(v) = (1 - v^2)^{-\frac{1}{2}}\phi_\pm(v, 1) + v^2\chi_\pm(v, 1, v)$$

$$\chi_\pm(\alpha, \beta, v) = \int_\alpha^\beta (s^2 - v^2)^{-\frac{3}{2}} \phi_\pm(v, s)ds$$

$$\Phi_\pm(v, s) = f(y \pm R, t - tv) - f(y \pm R, t - ts)$$

Expression (2.1), regularized in Marchaut form, also exists for weaker constraints on the function f, which were indicated above. Nevertheless, it was possible to extend the region of definition of the operator B, which includes the class of Hölder functions of order $\alpha > \frac{1}{2}$.

3. APPROXIMATE CALCULATION OF THE INTEGRO-DIFFERENTIAL EXPRESSION (1.1)

We will assume, for simplicity, that the function $f(\cdot, t)$ at the initial instant of time $t = 0$ vanishes. We will introduce the following notation

$$I_0(\alpha, \beta) = H(ct - x)\int_\alpha^\beta v^2\Psi_+(v)\chi_-(v, I, v)dv$$

$$I_1(\alpha, \beta, \gamma, \delta) = H(ct - x)\int_\alpha^\beta v^2\Psi_+(v)\chi_-(\gamma, \delta, v)dv$$

$$I_2(\alpha, \beta) = H(ct - x)\int_\alpha^\beta \Psi_+(v)dv$$

$$I_3(\alpha, \beta, \gamma) = H(ct - x)\int_\alpha^\beta v^2\Psi_+(v)\chi_-(v, \gamma, v)dv$$

To approximate the right-hand side of (2.1) it is sufficient to construct approximate formulae for evaluating the integrals $I_0(v_0, w)$ and $I_2(v_0, w)$. Hence, the minus sign on the function $\phi_-(v, s)$ will henceforth be omitted

We will assume that the function f is defined by the matrix $\{f(n, i)\}$ of its values $f(y_m, t_i)$ at the points $y_m = (m - \frac{1}{2})h$ $(m = 1, 2, \ldots, M)$, $t_i = i\tau$ $(i = 1, 2, \ldots, J)$ (h and τ are the steps with respect to the space and time variables, respectively), while the function $g(x, \alpha)$ is specified analytically. Approximate values of $I_0(v_0, w)$ and $I_2(v_0, w)$ will be sought in the same pattern (i.e. at the points y_m and t_i).

We will introduce an integer index N which we will put equal to $m - 1$, if $x^2 + y^2 \leq c^2t^2$. Otherwise $(x^2 + y^2 \geq t^2)$ we will assume $N = [\sqrt{(c^2t^2 - x^2)}/h]$ is the integer part of the number $\sqrt{(c^2t^2 - x^2)}/h$. We will also introduce the sequence of points $v_n = \sqrt{(n^2t^2 + x^2)}/(ct)$ $(n = 0, 1, \ldots, N)$ and $s_v = 1 - v/j$ $(v = 0, 1, \ldots, in)$, where

$$\text{in} = [j - \sqrt{n^2h^2 + x^2}/(c\tau)] \tag{3.1}$$

is the integer part of the number in the square brackets. If $\text{in} \leqslant 0$, the sequence of points $\{s_v\}$ is ignored.

For $N \geqslant 1$ we can represent the expressions for $I_0(v_0, w)$ and $I_2(v_0, w)$ in the form of sums

$$I_k(v_0, w) = I_k(v_N, w) + \sum_{n=1}^{N} I_k(v_{n-1}, v_n) \; (k = 0, 2)$$

If the indices "iN" or "in" are greater than zero, we have

$$I_0(v_N, w) = I_3(v_N, w, s_{\text{iN}}) + \sum_{v=1}^{\text{iN}} I_1(v_N, w, s_v, s_{v-1})$$

$$I_0(v_{n-1}, v_n) = I_3(v_{n-1}, v_n, s_{\text{iN}}) + \sum_{v=1}^{\text{in}} I_1(v_{n-1}, v_n, s_v, s_{v-1})$$

Approximate values of the integrals $I_1(v_{n-1}, v_n, s_v, s_{v-1})$ $(v = 1, 2, \ldots, \text{in};\ n = 1, 2, \ldots, N)$ can be found using well-known numerical methods, since the integral functions corresponding to them have no singularities.

To approximate the integrals $I_3(v_N, w, s_{\text{in}})$ and $I_3(v_{n-1}, v_n, s_{\text{in}})$ one can use the linear spline approximation

$$\phi(v, s) = \phi(v, s_{\text{in}})(s_{\text{in}} - v)^{-1}(s - v) + r(v, s) \tag{3.2}$$

where

$$r(v, s) = \tfrac{1}{2}\phi''_{ss}(\theta_s)(s - v)^2 - \tfrac{1}{2}\phi''_{ss}(\theta_{s\text{in}})(s_{\text{in}} - v)(s - v)$$

and θ_s and $\theta_{s_{\text{in}}}$ are certain points from the interval $[v\ s_{\text{in}}]$. As a result we obtain the following equations

$$I_3(v_N, w, s_{\text{iN}}) = \int_{v_N}^{w} \frac{ctv^2\phi(v, s_{\text{iN}})}{\sqrt{v^2 - v_0^2}\sqrt{s_{\text{iN}}^2 - v^2}}\, dv + R_{N0}(\phi) \tag{3.3}$$

$$I_3(v_{n-1}, v_n, s_{\text{in}}) = \int_{v_{n-1}}^{v_n} \frac{ctv^2\phi(v, s_{\text{in}})}{\sqrt{v^2 - v_0^2}\sqrt{s_{\text{in}}^2 - v^2}}\, dv + R_{n0}(\phi) \tag{3.4}$$

the errors of which $R_{N0}(\phi)$ and $R_{n0}(\phi)$ are of the order of $H(H+T)^{3/2}$, where $T = \tau/t = 1/j$, $H = h/(ct)$.

For $n > 1$ the positive function on the right-hand side of (3.4) has no singularities, and hence no difficulties arise in approximating the integral $I_3(v_{n-1}, v_n, s_{\text{iN}})$ in this case. If $n = 1$, we can replace the function $ctv^2\phi(v, s_{\text{in}})(s_{\text{in}} - v)^{-1/2}$ in the interval $[v_0, v_l]$ by a linear spline. As a result we obtain the following approximate formula

$$I_3(v_0, v_1, s_{i1}) = \left[\frac{v_0\varphi(v_0, s_{i1})}{\sqrt{s_{i1}^2 - v_0^2}} - \frac{v_1\varphi(v_1, s_{i1})}{\sqrt{s_{i1}^2 - v_1^2}}\right] \times \left[\frac{ctv_1v_0}{v_1 - v_0}\ln\frac{v_1 + \sqrt{v_1^2 - v_0^2}}{v_0} - \frac{h}{v_1 - v_0}\right] \tag{3.5}$$

For the approximate evaluation of the integral on the right-hand side of (3.3) one can use the formula of left rectangles. One can similarly obtain the approximate value of the integral $I_1(v_N, w, s_v, s_{v-1})$.

If the indices "iN" or "in" are less than unity, the integrals $I_0(v_N, w)$ or $I_0(v_{n-1}, v_n)$ can be approximated directly. We finally obtain

$$I_0(v_N, w) = \frac{(w - v_{m-1})v_{m-1}^2}{\sqrt{v_N^2 - v_0^2}} J_{m-1}(\phi(v, 1)),\quad w = \frac{\sqrt{x^2 + y^2}}{ct} < 1$$

$$I_0(v_N, w) = -\frac{ctv_N^2\phi(v_N, 1)}{\sqrt{v_N^2 - v_0^2}}\ln(1 + \sqrt{1 - v_N^2}),\quad \sqrt{x^2 + y^2} > ct$$

$$I_0(\nu_{n-1}, \nu_n) = \tfrac{1}{2}(\nu_n - \nu_{n-1})\left[\frac{\nu_n^2 J_n(\phi(\nu, 1))}{\nu_n^2 - \nu_0^2} + \frac{\nu_{n-1}^2 J_{n-1}(\phi(\nu, 1))}{\sqrt{\nu_{n-1}^2 - \nu_0^2}}\right]$$

$$J_n(f(\nu)) = \frac{ctf(\nu_n)}{\sqrt{1 - \nu_n^2}}$$

The approximate values of the integral $I_0(\nu_0, \nu_1)$ are found from (3.5) where we must replace the number s_{ii} by unity.

For the approximate evaluation of the integral $I_2(\nu_N, w)$ we replace the integrand without the factor $\sqrt{(1-\nu^2)}$ by its value at the left-hand end of the integral.

The integral $I_2(\nu_{n-1}, \nu_n)$, when $n>0$, is approximated using the trapezoidal rule.

If $n=0$, integrating the function $ct\nu/\sqrt{(1-\nu^2)}f(\nu-R, t-t\nu)g(x, R)$ by the linear spline in the interval $[\nu_0, \nu_1]$ we obtain the approximate formula

$$I_2(\nu_0, \nu_1) = H(ct-x)\frac{1}{\nu_1 - \nu_0}\left\{g(x, mh)J_0(f(m, t-t\nu))\left[H - \nu_0 \ln\frac{\nu_1 + H}{\nu_0}\right] - \right.$$
$$\left. - g(x, (m-1)h)J_1(f(m-1, t-t\nu))\left[H - \nu_1 \ln\frac{\nu_1 + H}{\nu_0}\right]\right.$$

The approximation equations given above contain values of the function $f(\ldots, t-t\nu_n)$ at points which do not belong to the pattern. Using the index (3.1) we can obtain values of the function $f(\ldots, t-t\nu_n)$ by linear interpolation

$$f(\ldots, t - t\nu_n) = (1 + \text{in} - j + j\nu_n)f(\ldots, \text{in}) + (j - j\nu_n - \text{in})f(\ldots, \text{in} + 1) \tag{3.6}$$

We will consider, finally, the case when $N=0$. Suppose $x^2+y^2<c^2t^2$ and $m=1$. In the interval $[\nu_0, \sqrt{(x^2+y^2)}/(ct)]$ we replace the function $(1-\nu^2)^{-1/2}g(x, R)\times f(y-R, t-t\nu)$ by its value at the left-hand end of this interval. As a result we obtain the following approximate formula

$$I_2(\nu_0, w) = H(ct-x)\frac{yct}{\sqrt{c^2t^2 - x^2}}\, g(x, h) f(1, t - t\nu_0)$$

The function $f(1, t-t\nu_0)$ must be determined by linear interpolation (3.6).

The approximate evaluation of the integral $I_0(\nu_0, w)$ in the case when $m=1$ and $ct \geqslant \sqrt{(x^2+y^2)}$ depends on the value of the index ii, defined as the integer part of the number $j-\sqrt{(x^2+y^2)}/(c\tau)$. If $\text{ii} \geqslant 1$, we have

$$I_0(\nu_0, w) = I_3(\nu_0, w, s_{\text{ii}}) + \sum_{\nu=2}^{\text{ii}} I_1(\nu_0, w, s_\nu, s_{\nu-1})$$

To approximate $I_3(\nu_0, w, s_{\text{ii}})$ we interpolate the function $\phi(\nu, \cdot)$ by a linear spline (3.2). Then

$$I_3(\nu_0, w, s_{ii}) = \int_{\nu_0}^{\sqrt{x^2+y^2}/(ct)} \frac{ct\nu^2 g(x, R)}{\sqrt{\nu^2 - \nu_0^2}\sqrt{s_{ii}^2 - \nu^2}}\, \phi(\nu, s_{ii})d\nu$$

Replacing the function $\nu g(x, R)\times(s_{ii}^2-\nu^2)^{-1/2}\phi(\nu, s_{ii})$ in the interval $[\nu_0, \sqrt{(x^2+y^2)}/(ct)]$ by its value at the point $\nu=\nu_0$, we obtain

$$I_3(\nu_0, w, s_{ii}) = \frac{xyg(x, 0)}{\text{ct}\sqrt{s_{ii}^2 - \nu_0^2}}\, \varphi(\nu_0, s_{ii})$$

We can similarly find the approximate values of the integrals $I_i(\nu_0, w, s_\nu, s_{\nu-1})$ $(\nu = 1, 2, \ldots, \text{ii})$.

If $\text{ii} \leqslant 0$, the use of linear interpolation (3.2) with respect to the variable s and modification of the left-rectangle formula leads to the following approximate equation

$$I_3(v_0, w, 1) = H(ct - x)\frac{xyg(x, 0)}{\sqrt{c^2t^2 - x^2}} f(m, t - tv_0)$$

Suppose $\sqrt{(x^2+y^2)} > ct$ and $N = 0$. Replacing the function $g(x, R)f(y-R, t-tv)$ by the constant $g(x, 0)f(m, t-tv_0)$, we will have

$$I_2(v_0, w) = H(ct - x)g(x, 0)f(m, t - tv_0)\pi/2$$

By analogy with what was done for the case when ii $\leqslant 0$, we can obtain an approximate value of the integral $I_0(v_0, w)$.

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